Fundamentals of Media Processing

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Course Overview (15 classes in total)

1-10 Machine Learning by Prof. Satoshi Ikehata

11-15 Signal Processing by Prof. Kazuya Kodama

Grading will be based on the final report.

10/16 (Today) Introduction Chap. 1

Basic of Machine Learning (Maybe for beginners)

10/23 Basic mathematics (1) (Linear algebra, probability, numerical computation) Chap. 2,3,4

10/30 Basic mathematics (2) (Linear algebra, probability, numerical computation) Chap. 2,3,4

11/6 Machine Learning Basics (1) Chap. 5

11/13 Machine Learning Basics (2) Chap. 5

Basic of Deep Learning

11/20 Deep Feedforward Networks Chap. 6

11/27 Regularization and Deep Learning Chap. 7

12/4 Optimization for Training Deep Models Chap. 8

CNN and its Application

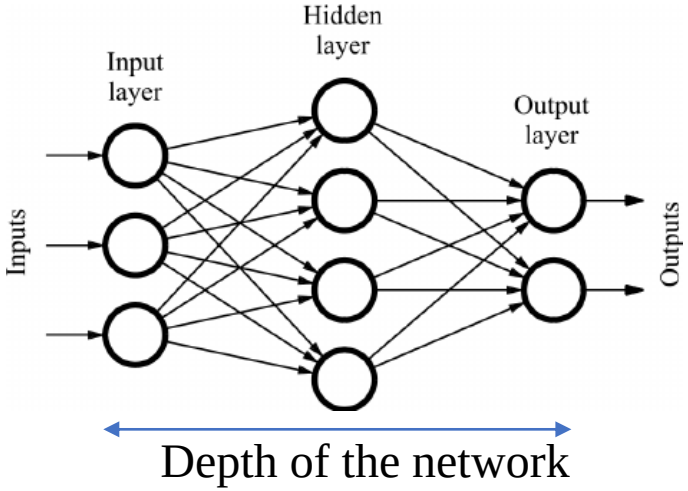
12/11 Convolutional Neural Networks and Its Application (1) Chap. 9 and more

12/18 Convolutional Neural Networks and Its Application (2) Chap. 9 and more

Deep Feedforward Networks

Deep Feedforward Networks

- Deep feedforward network (or multilayer perceptron; MLP) approximates a function $y = f^*(x)$ by $f(x;\theta)$ and learns θ
- Information flows through the function being evaluated from *x*, through the intermediate computation used to define *f* and finally to the output *y*. There are no feedbacks
- A simple linear model cannot learn. To get the nonlinearity of the function, we introduce the nonlinear transformation of x as $y = f(x;\theta,\boldsymbol{\omega}) = \phi(x;\theta)^T\boldsymbol{\omega}$**:** The deep neural networks try to learn ϕ



Learning XOR Network (1)

- Training from pairs of x_1, x_2 ([0,0], [0,1], [1,0], [1,1]), learn the function that returns $x_1 \oplus x_2$ by minimizing

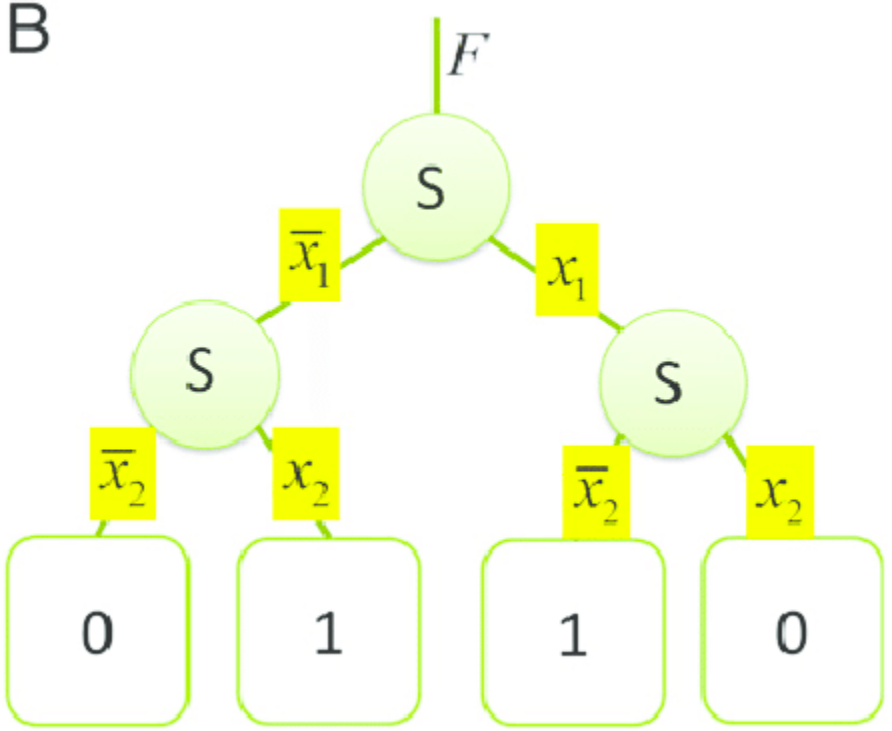
$$J(\boldsymbol{\theta}) = \frac{1}{4}\sum_{\boldsymbol{x}\in\mathbb{X}} (f^*(\boldsymbol{x}) - f(\boldsymbol{x};\boldsymbol{\theta}))^2$$

A

$$F(x_1, x_2) = x_1 \oplus x_2$$

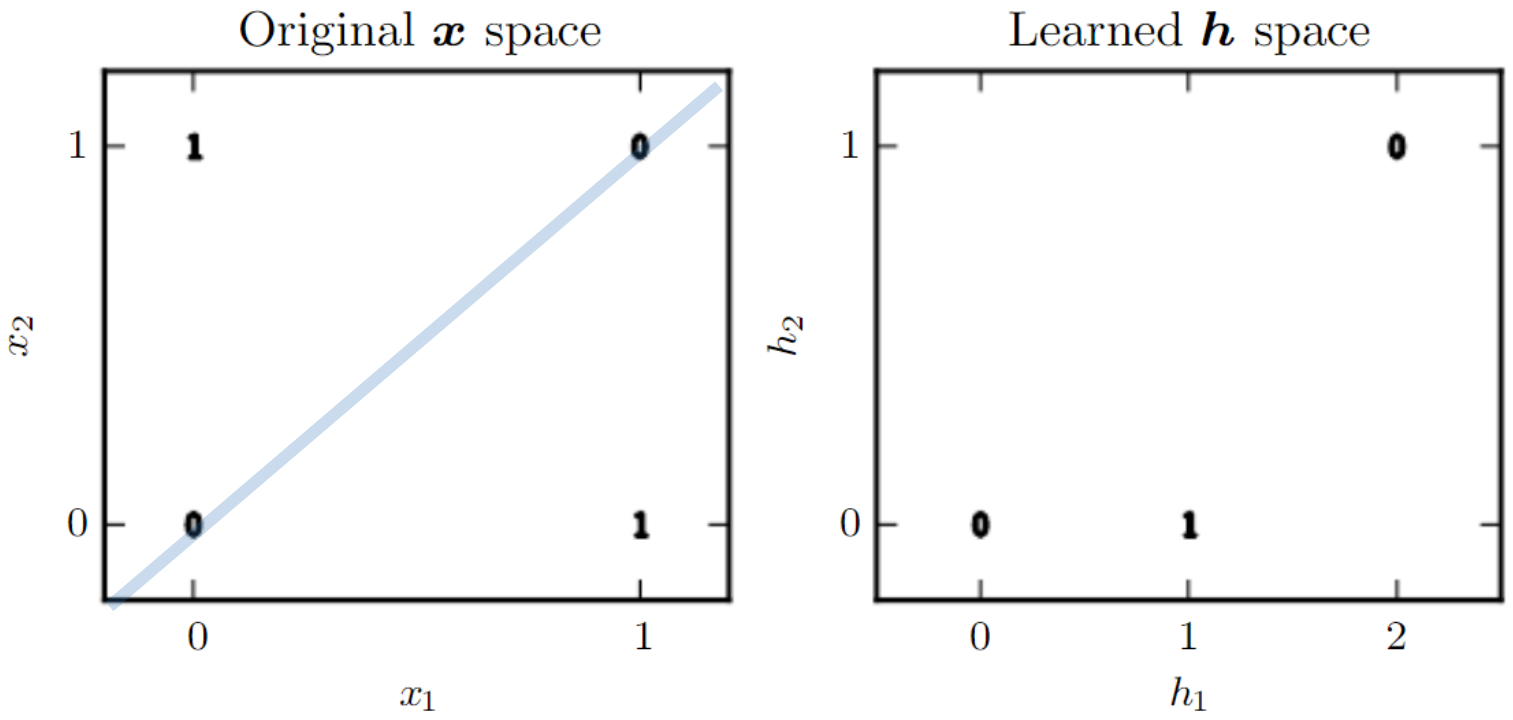
$x=(x_1,x_2)$	$F(x)$
00	0
01	1
10	1
11	0

B



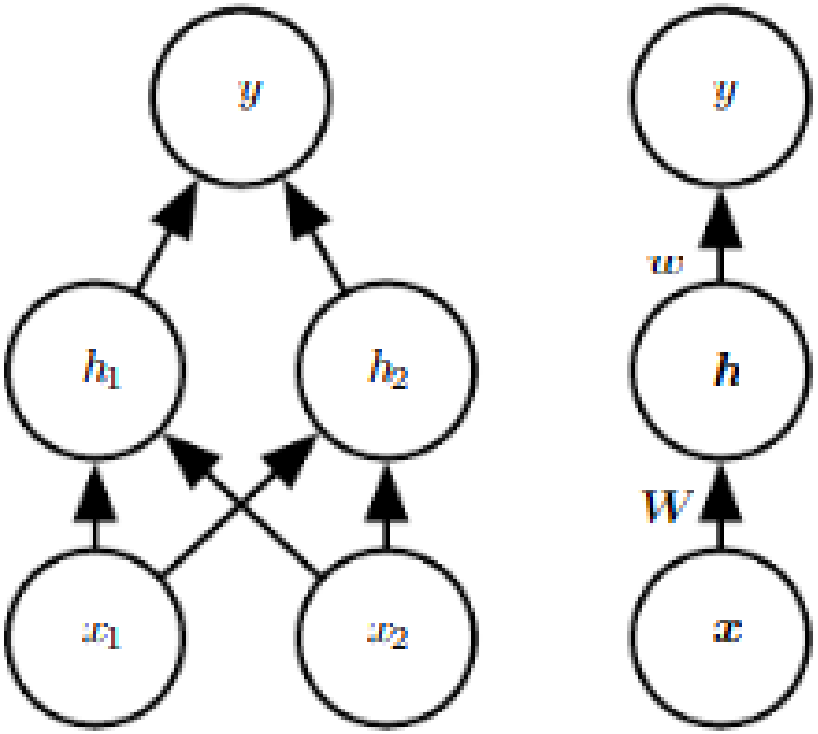
Learning XOR Network (2)

- Simple linear model ($y = \boldsymbol{\omega}^T \boldsymbol{x} + b$) cannot solve this problem, since the problem is not linearly separable



Learning XOR Network (3)

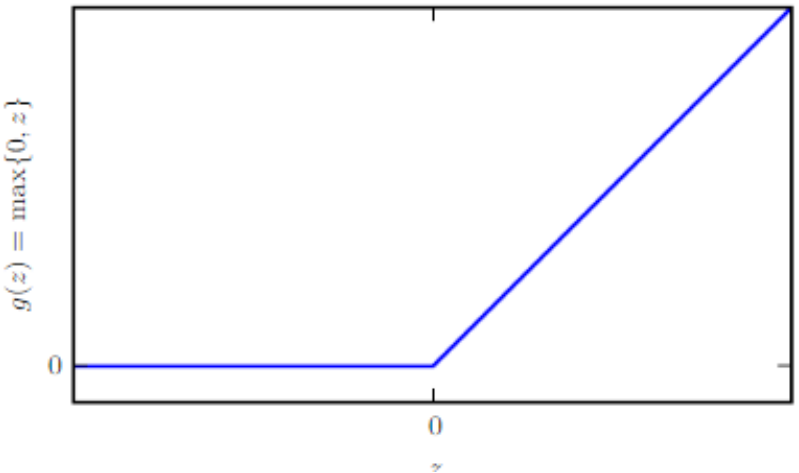
- Consider a simple feedforward network:
  - $\boldsymbol{h} = f^1(\boldsymbol{x}; W, \boldsymbol{c}); W^T\boldsymbol{x} + \boldsymbol{c}$
  - $y = f^2(h; \boldsymbol{\omega}, b); \boldsymbol{\omega}^T\boldsymbol{h} + b$
  - $y = f^2\left(f^1(x)\right) = f(x; W, \boldsymbol{\omega}, b, \boldsymbol{c}) = \boldsymbol{x}^T W\boldsymbol{\omega} = \boldsymbol{x}^T\boldsymbol{\omega}'$



Learning XOR Network (4)

- Since the linear function cannot solve the problem, we need some "nonlinearity" in the function $y = f^2\left(f^1(x)\right)$
  - $\boldsymbol{h} = W^T\boldsymbol{x} + \boldsymbol{c} \rightarrow \boldsymbol{h} = g(W^T\boldsymbol{x} + \boldsymbol{c})$: g is a nonlinear function

- In modern neural networks, the default recommendation of g is to use the ***rectified linear unit*** (ReLU; Jarrettet al., 2009), defined by the activation function g(z) = max{0, z}.



Learning XOR Network (5)

- We can now specify our complete feedforward model as
  - $f(x; W, \boldsymbol{\omega}, b, \boldsymbol{c}) = \boldsymbol{\omega}^T \max(0, W^T \boldsymbol{x} + \boldsymbol{c}) + b$

$$\boldsymbol{X} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\boldsymbol{W} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\boldsymbol{c} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \qquad b = 0$$

$$\boldsymbol{w} = \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

Training result

$$\boldsymbol{XW} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 2 & 2 \end{bmatrix} \quad \underset{\rightarrow}{XW + \boldsymbol{c}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} \quad \underset{\rightarrow}{ReLu} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} \quad \underset{\rightarrow}{\boldsymbol{\omega}, b} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Gradient-based Learning

- Cost function in neural network is generally non-convex, therefore iterative (stochastic, gradient-based) optimization is required

- All weights are initialized by small random values, biases are initialized by small positive values or zeros

- ***Back-propagation*** (BackProp) algorithm, which is common in deep neural networks, is used to compute the gradient

Cost Function

- Most modern deep neural networks are trained using the *maximum likelihood estimation*. Negative log-likelihood helps to avoid the gradient of the cost function being too small when its argument is too negative (i.e., $\frac{\mathrm{d}e^{\|y-y\|^2}}{dx} \sim e^x, \frac{\mathrm{d}\log(e^x)}{dx} \sim x$)

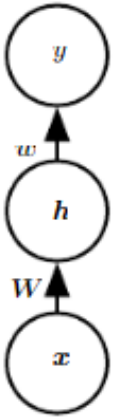
$$J(\boldsymbol{\theta}) = -\mathbb{E}_{\mathbf{x},\mathbf{y}\sim\hat{p}_{\text{data}}} \log p_{\text{model}}(\boldsymbol{y} \mid \boldsymbol{x}).$$ Cross entropy loss

$$J(\theta) = \frac{1}{2}\mathbb{E}_{\mathbf{x},\mathbf{y}\sim\hat{p}_{\text{data}}} ||\boldsymbol{y} - f(\boldsymbol{x};\boldsymbol{\theta})||^2 + \text{const}$$ When $p_{\text{model}} = \mathcal{N}(\boldsymbol{y}; f(\boldsymbol{x};\boldsymbol{\theta}); I)$

- Cross-entropy cost function is more popular than mean-squared error or mean absolute error since some output units that saturate produce very small gradients when combined with these cost functions

Output Units (1)

- ***Linear unit***: Given features $\boldsymbol{h}$, it outputs as
  - $\hat{y} = W^T \boldsymbol{h} + b$
  - Often used to produce a mean of a conditional Gaussian distribution



- ***Sigmoid unit***: Given features $\boldsymbol{h}$, it outputs as
  - $\hat{y} = \sigma(W^T \boldsymbol{h} + b)$: σ is a logistic sigmoid function
  - Binary classification problem needs to predict the probability P(y=1|x) within the range of [0,1]
  - We can not define gradient outside the intervals of [0,1] when we simply use

$$P(y = 1 \mid \boldsymbol{x}) = \max\left\{0, \min\left\{1, \boldsymbol{w}^\top \boldsymbol{h} + b\right\}\right\}$$

  - Derivative function of sigmoid is computed anywhere

Output Units (2)

- ***Softmax unit***: Output unit for multilabel classification problem
  - Multilabel classification problem needs to predict the probability $\hat{y}_i = P(y = i|\boldsymbol{x})$ within the range of [0,1] and summation of $\hat{y}_i$ must be one
  - Works well with the log-likelihood-based loss, but many other objective functions other than log-likelihood does not work because the gradient will vanish when the argument to exp becomes very negative
  - Named "softmax" because it is continuous and differentiable version of max

$$\underset{i = 1,\ldots,\#\text{ of class}}{\text{softmax}(\boldsymbol{z})_i} = \frac{\exp(z_i)}{\sum_j \exp(z_j)}. \qquad \boldsymbol{z} = \boldsymbol{W}^\top \boldsymbol{h} + \boldsymbol{b},$$

Hidden Units (1)

- ***Rectified linear units*** $g(z) = \max\{0, z\}$ are a default choice of hidden unit but one drawbacks of ReLU is they cannot learn via gradient-based methods on examples where their activation is zero
  - Nondifferentiable at z=0 isn't problematic in most cases

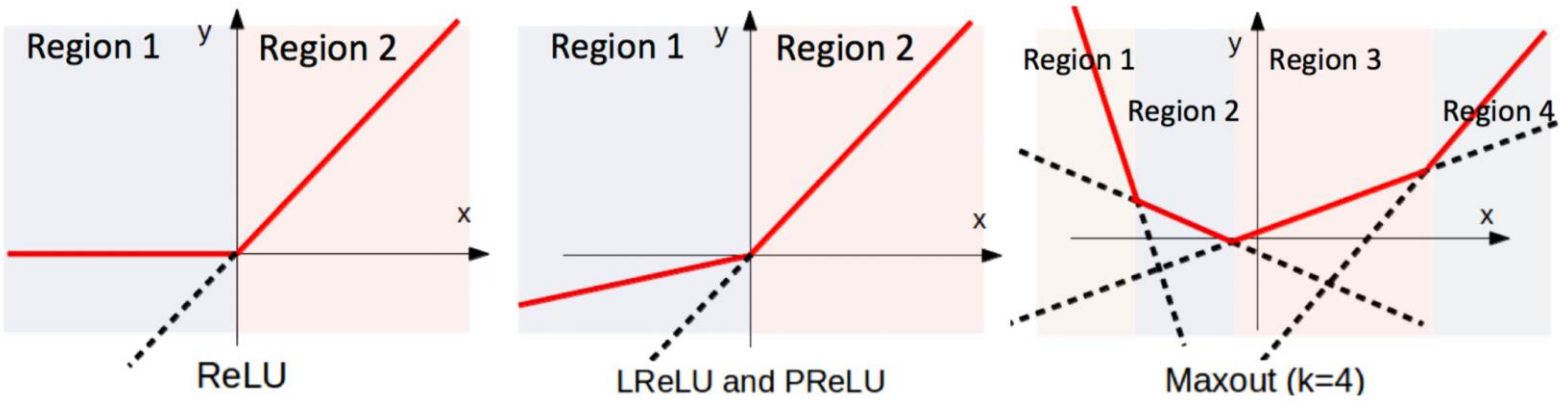
- Three variations are based on using a nonzero slope α_i when $z_i < 0$: $h_i = g(z, \alpha)_i = \max(0, z_i) + \alpha_i \min(0, z_i)$
  - ***Absolute value rectification*** : $\alpha_i = -1$ $(g(z) = |z|)$
    - Was used for object recognition from images (Jarrett 2009)
  - ***A leaky ReLU*** : α_i is a fixed small value like 0.01 (Maas2013)
  - ***A parametric ReLU (PReLU)leaky ReLU*** : α_i is a learnable parameter (He2015)

Hidden Units (2)

- ***Maxout units:***
  - Instead of applying an element-wise function, maxout units divide ***z*** into groups of *k* values. Each maxout unit then outputs the maximum element of one of these groups which provides a way of learning a piecewise linear function that responds to multiple directions in the input ***x*** space

$$u_{ik}^{(l)} = \sum_{j=1}^{m} (w_{ijk}^{(l)} z_j^{(l-1)}) + b_{ik}^{(l)}$$

$$z_i^{(l)} = max(u_{ik}^{(l)})$$



https://cs.adelaide.edu.au/~carneiro/publications/226.pdf

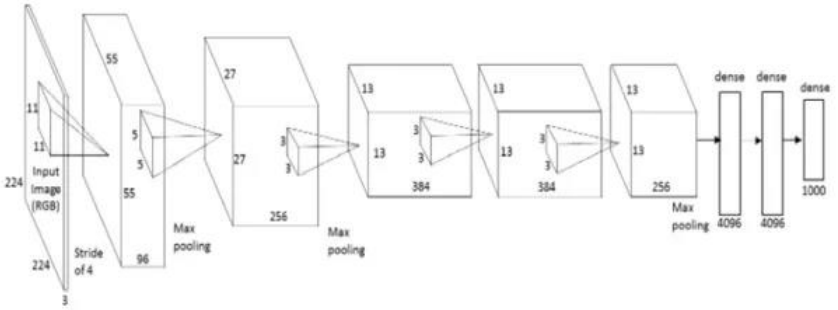
Hidden Units (3)

- ***Logistic sigmoid, hyperbolic tangent:*** $g(z) = \sigma(z),\ \tanh(z)$
  - Activation functions prior to the ReLU
  - They saturate to a high value when *z* is very positive, saturate to a low value when *z* is very negative, therefore a gradient-based algorithm was very difficult
  - Therefore, rather used as an output unit

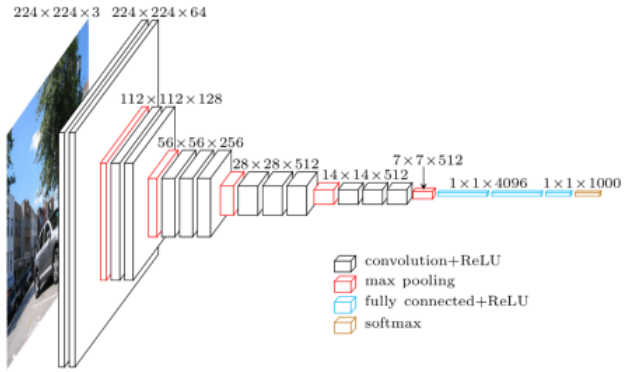
- Other hidden units (not commonly used)
  - Radial basis function unit
  - Softplus ($g(a) = \log(1 + e^a)$)
  - Hard tanh ($g(a) = \max(-1, \min(1, a))$)
  - Etc…

Architecture Design (1)

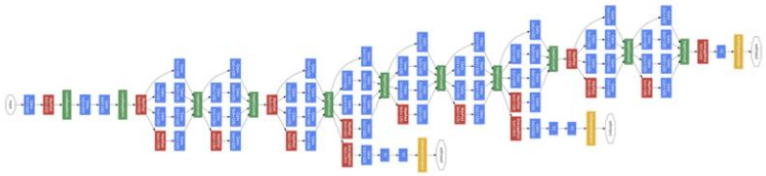
- ***Architecture*** refers to the overall structure of the network
  - Most neural networks are organized into groups of units called ***layers*** (i.e., unit ≠ layer)



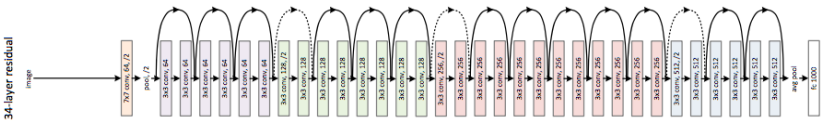
Alex-Net (Krizhevsky2012)
8 layers



VGG-Net (Simonyan2013)
16 layers



Google-Net (Segedy2015)
22 layers



Res-Net (He2015)
125 layers

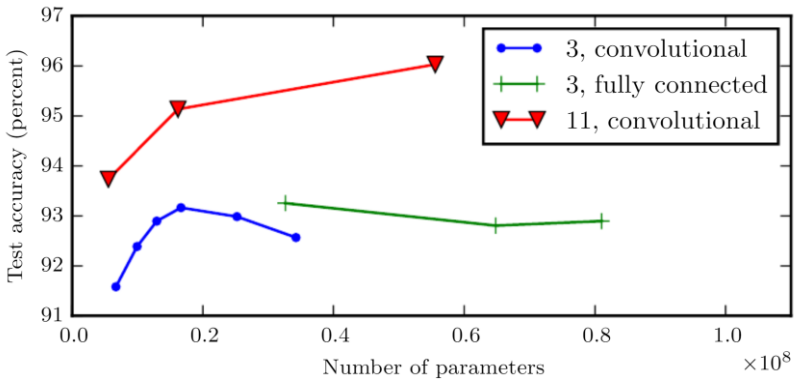
Convolution
Pooling
Softmax
Other

Architecture Design (2)

- ***Universal Approximation theorem*** (Hornik1989)

  - Regardless of what function we are trying to learn, a feedforward network with a single layer is sufficient to represent any function. However, the layer may be infeasibly large and may fail to learn and generalize correctly

  - In many circumstances, using deeper models can reduce the number of units required to represent the desired function and can reduce the amount of generalization error



Architecture Design (3)

- ***Skip connections***
  - Going from layer i to layer $i + 2$ or higher, which makes it easier for the gradient to flow from output layers to layers nearer the input (e.g., ResNet)

- ***Dense connections***
  - Going from layer i, layer $i + 1$ to layer $i + 2$, which reduces the number of parameters to represent the function (e.g., DenseNet, Huan2017)

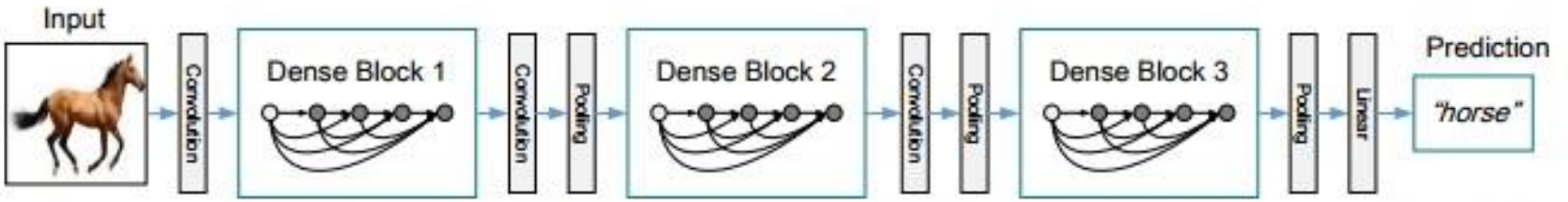
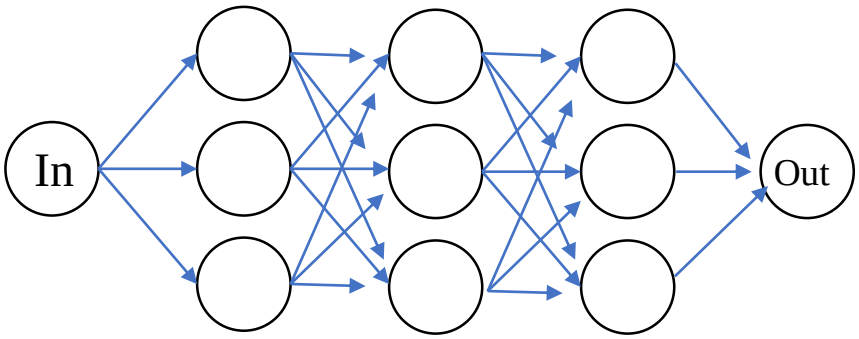


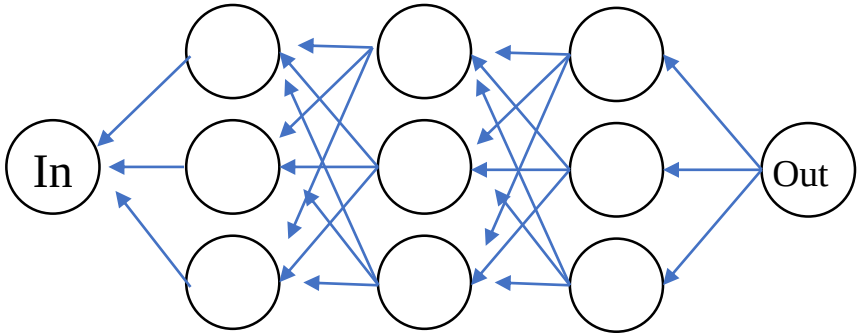
Figure 2. A deep DenseNet with three dense blocks. The layers between two adjacent blocks are referred to as transition layers and change feature map sizes via convolution and pooling.

Back Propagation (1)

- ***Forward propagation***
  - Given input x, the information is propagated up to the hidden units at each layer and finally an output $\hat{y}$ is produced

- ***Back propagation*** (***backprop***; Rumelhart1986)
  - The back propagation algorithm allows information from the cost to then flow backward through the network (or graph) in order *to* compute the gradient of a function (*Not specific for the deep learning algorithm*)



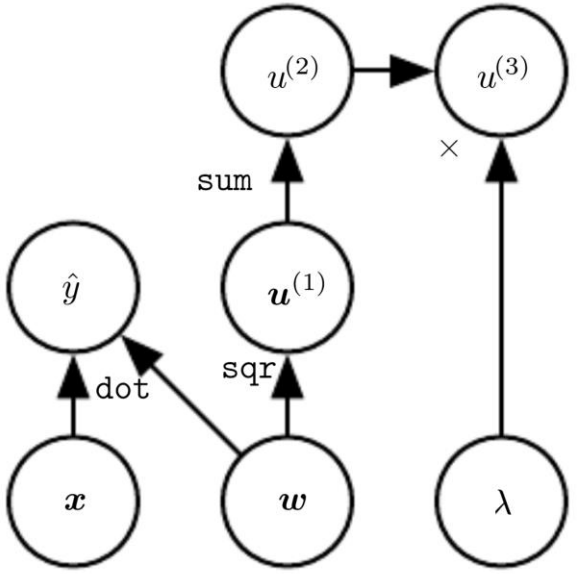


Back Propagation (2)

- ***Computational Graph***
  - Each ***node*** in the graph indicates a variable
  - An ***operation*** is a simple function of one or more variables
    - If a variable y is computed by applying an operation to a variable x, then we draw a directed edge from x to y

- Notation about Gradient
  - $\nabla_{\mathbf{X}} z$ denotes the gradient of value z with respect to a tensor $\mathbf{X}$
  - $(\nabla_{\mathbf{X}} z)_i$ gives $\partial z / \partial x_i$
  - If $\mathbf{Y} = g(\mathbf{X})$ and $z = f(\mathbf{Y})$, then

$$\nabla_{\mathbf{X}} z = \sum_j (\nabla_{\mathbf{X}} Y_j) \frac{\partial z}{\partial Y_i} \quad \text{(Chain rule)}$$



Back Propagation (3)

- First, consider a computational graph describing how to compute a single scalar $u^{(n)}$ (i.e., the loss on a training example)
- We want to obtain gradient with respect to the n_i input nodes $(\frac{\partial u^{(n)}}{\partial u^{(i)}}, i \in \{1,2,\dots,n_i\})$
- Each node w.r.t hidden units $i \in \{n_{i+1},\dots,n_{n-1}\}$ is associated with an operator $f^{(i)}$ and is computed by evaluating the function $u^{(i)} = f\left(\mathbb{A}^{(i)}\right)$, where $\mathbb{A}^{(i)}$ is the set of all nodes that are parents of $u^{(i)}$

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for $i = 1, \dots, n_i$ **do**
  $u^{(i)} \leftarrow x_i$
end for
for $i = n_i + 1, \dots, n$ **do**
  $\mathbb{A}^{(i)} \leftarrow \{u^{(j)} \mid j \in Pa(u^{(i)})\}$
  $u^{(i)} \leftarrow f^{(i)}(\mathbb{A}^{(i)})$
end for
return $u^{(n)}$

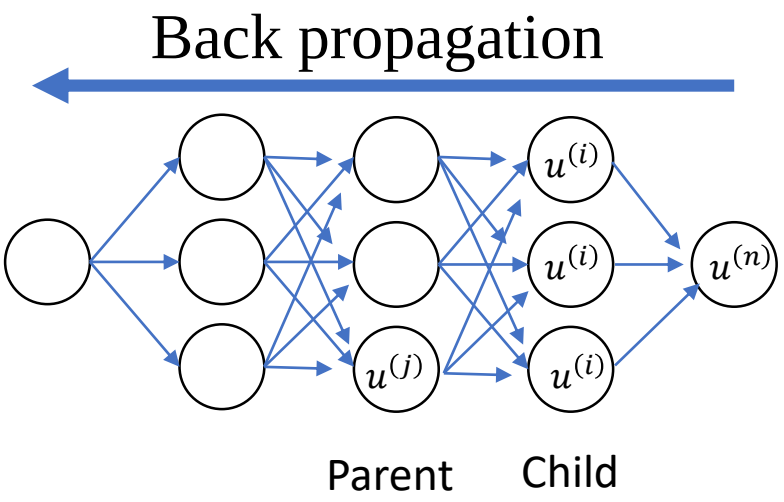
Algorithm of forward propagation

Back Propagation (4)

- We denote the computational subgraph $\mathcal{B}$ for backprop with one node per node of graph $\mathcal{G}$ for forward prop. Each node of $\mathcal{B}$ computes the derivative $\partial u^{(n)}/\partial u^{(i)}$ via:

$$\frac{\partial u^{(n)}}{\partial u^{(j)}} = \sum_{i:j \in Parents(u^{(i)})} \frac{\partial u^{(n)}}{\partial u^{(i)}} \frac{\partial u^{(i)}}{\partial u^{(j)}}$$

Run forward propagation (algorithm 6.1 for this example) to obtain the activations of the network.
Initialize `grad_table`, a data structure that will store the derivatives that have been computed. The entry `grad_table`$[u^{(i)}]$ will store the computed value of $\frac{\partial u^{(n)}}{\partial u^{(i)}}$.
`grad_table`$[u^{(n)}] \leftarrow 1$
for $j = n-1$ down to 1 **do**
  The next line computes $\frac{\partial u^{(n)}}{\partial u^{(j)}} = \sum_{i:j \in Pa(u^{(i)})} \frac{\partial u^{(n)}}{\partial u^{(i)}} \frac{\partial u^{(i)}}{\partial u^{(j)}}$ using stored values:
  `grad_table`$[u^{(j)}] \leftarrow \sum_{i:j \in Pa(u^{(i)})}$ `grad_table`$[u^{(i)}] \frac{\partial u^{(i)}}{\partial u^{(j)}}$
end for
return {`grad_table`$[u^{(i)}] \mid i = 1, \ldots, n_i$}



Back Propagation (5)

- Example: a fully connected MLP (multi-layer perceptron)

Algorithm: Forward Propagation

Require: Network depth, l
Require: $\boldsymbol{W}^{(i)}, i \in \{1, \ldots, l\}$, the weight matrices of the model
Require: $\boldsymbol{b}^{(i)}, i \in \{1, \ldots, l\}$, the bias parameters of the model
Require: $\boldsymbol{x}$, the input to process
Require: $\boldsymbol{y}$, the target output

$\boldsymbol{h}^{(0)} = \boldsymbol{x}$
for $k = 1, \ldots, l$ **do**
  $\boldsymbol{a}^{(k)} = \boldsymbol{b}^{(k)} + \boldsymbol{W}^{(k)} \boldsymbol{h}^{(k-1)}$
  $\boldsymbol{h}^{(k)} = f(\boldsymbol{a}^{(k)})$
end for
$\hat{\boldsymbol{y}} = \boldsymbol{h}^{(l)}$
$J = L(\hat{\boldsymbol{y}}, \boldsymbol{y}) + \lambda \Omega(\theta)$

Back Propagation (6)

$$\boldsymbol{a}^{(k)} = \boldsymbol{b}^{(k)} + \boldsymbol{W}^{(k)}\boldsymbol{h}^{(k-1)}$$
$$\boldsymbol{h}^{(k)} = f(\boldsymbol{a}^{(k)})$$

Algorithm: Back Propagation

After the forward computation, compute the gradient on the output layer:

$\boldsymbol{g} \leftarrow \nabla_{\hat{\boldsymbol{y}}} J = \nabla_{\hat{\boldsymbol{y}}} L(\hat{\boldsymbol{y}}, \boldsymbol{y})$

for $k = l, l-1, \ldots, 1$ **do**

Convert the gradient on the layer's output into a gradient on the pre-nonlinearity activation (element-wise multiplication if f is element-wise):

$\boldsymbol{g} \leftarrow \nabla_{\boldsymbol{a}^{(k)}} J = \boldsymbol{g} \odot f'(\boldsymbol{a}^{(k)})$ e.g., $\frac{\partial J}{\partial \boldsymbol{a}^k} = \frac{\partial J}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial \boldsymbol{a}^k} = \frac{\partial L}{\partial \hat{y}} \cdot f'(\boldsymbol{a}^k)$

Compute gradients on weights and biases (including the regularization term, where needed):

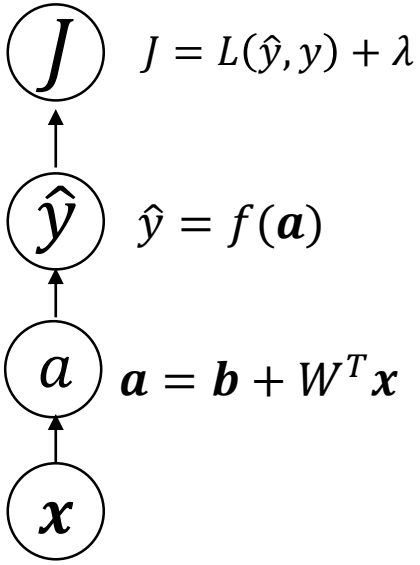
$\nabla_{\boldsymbol{b}^{(k)}} J = \boldsymbol{g} + \lambda \nabla_{\boldsymbol{b}^{(k)}} \Omega(\theta)$ e.g., $\frac{\partial J}{\partial b^k} = \frac{\partial J}{\partial \boldsymbol{a}^k} \frac{\partial \boldsymbol{a}^k}{\partial \boldsymbol{b}^k} = \frac{\partial J}{\partial \boldsymbol{a}^k} + (\boldsymbol{reg}.)$

$\nabla_{\boldsymbol{W}^{(k)}} J = \boldsymbol{g}\, \boldsymbol{h}^{(k-1)\top} + \lambda \nabla_{\boldsymbol{W}^{(k)}} \Omega(\theta)$ e.g., $\frac{\partial J}{\partial W^k} = \frac{\partial J}{\partial \boldsymbol{a}^k} \frac{\partial \boldsymbol{a}^k}{\partial W} = \frac{\partial J}{\partial \boldsymbol{a}^k} \boldsymbol{h}^{k^T} + (\boldsymbol{reg}.)$

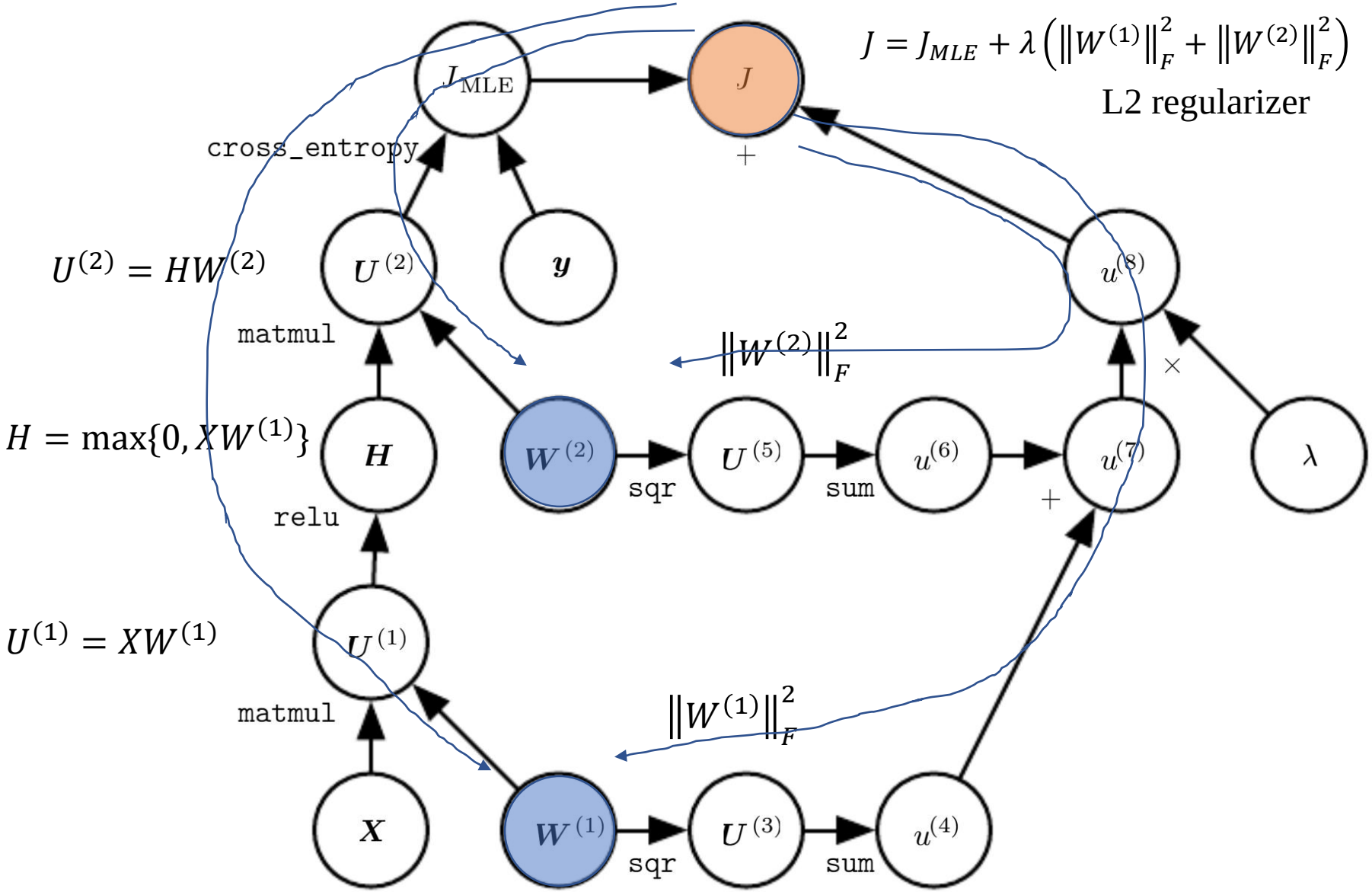
Propagate the gradients w.r.t. the next lower-level hidden layer's activations:

$\boldsymbol{g} \leftarrow \nabla_{\boldsymbol{h}^{(k-1)}} J = \boldsymbol{W}^{(k)\top}\, \boldsymbol{g}$ e.g., $\frac{\partial J}{\partial \boldsymbol{h}^{k-1}} = \frac{\partial J}{\partial \boldsymbol{a}^k} \frac{\partial \boldsymbol{a}^k}{\partial \boldsymbol{h}^{k-1}} = W^T \frac{\partial J}{\partial \boldsymbol{a}^{k-1}}$

end for



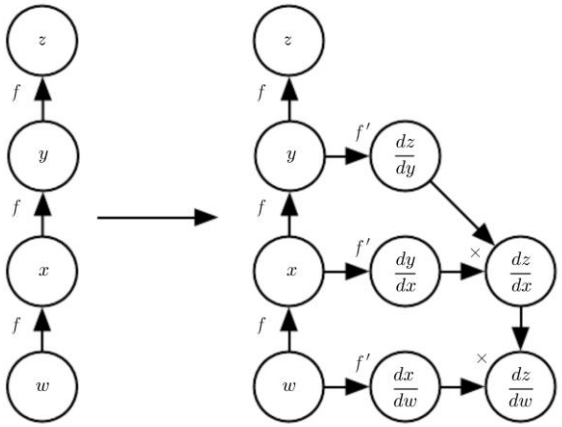
Back Propagation (7)



Implementation of General Back-Propagation (1)

- ***Symbolic-to-number differentiation*** (Used in Torch and Caffe)
  - Take a computational graph and as set of numerical values for the inputs to the graph, then return a set of numerical values describing the gradient at those input values

- ***Symbolic-to-symbol differentiation***
  (Used in Theano and TensorFlow)
  - Take a computational graph and add additional nodes to the graph that provides a symbolic description of the desired derivatives.
  - It is possible to run backpropagation again, differentiating the derivatives to obtain higher derivatives (e.g., for computing Hessian)



Implementation of General Back-Propagation (2)

- Software implementations of backprop provide both the operations and their "bprop" method

- We assume that each variable $\mathbf{V}$ associated with assumptions:
  - get_operation($\mathbf{V}$): Returns the operation that computes $\mathbf{V}$, represented by the edges coming into $\mathbf{V}$ in the graph
  - get_consumers($\mathbf{V}, \mathcal{G}$): Returns the list of variables that are children of $\mathbf{V}$ in the computational graph $\mathcal{G}$
  - get_inputs($\mathbf{V}, \mathcal{G}$): Returns the list of variables that are parents of $\mathbf{V}$ in the computational graph $\mathcal{G}$
  - Each operation is associated with a "bprop" operation, which computes a Jacobian-vector product as $\nabla_{\mathbf{X}} z = \sum_j \left(\nabla_{\mathbf{X}} Y_j\right) \frac{\partial z}{\partial Y_i}$
  - For example, given a multiplication operation to create a variable $C = AB$, bprop requests the gradient w.r.t. A or B without knowing any differentiation rules.

Implementation of General Back-Propagation (3)

- When "bprop" is called, op.bprop(inputs, X, G) returns:

$$\sum_i (\nabla_X \text{op.f}(\text{inputs})_i) G_i$$

  - Here, inputs is a list of inputs that are supplied to the operation
  - op.f is the mathematical function that the operation implements
  - X is the input whose gradient we wish to compute
  - G is the gradient on the output of the operation

- *Software engineers who build a new implementation of back-propagation or advanced users who need to add their own operation to an existing library mush usually derive the op.bprop method for any new operations manually*

Implementation of General Back-Propagation (4)

- The deep learning community uses computational graphs that are usually represented by explicit data structures created by specialized libraries
- It requires the library developer to define the bprop methods for every operation and limiting the users of the library to only those operations that have been defined
- However it has benefit of allowing customized back-propagation rules to be developed for each operation, enabling the developer to improve speed or stability in nonobvious
- Back-propagation is not the only way of computing the gradient, but it is a practical method that continues to serve the deep learning community well

High-Order Derivatives

- We are often interested in computing the Hessian matrix.
  - If we have a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$, the Hessian matrix is of size $n \times n$
  - Since n will be the number of parameters, the entire Hessian matrix is infeasible to even present

- ***Krylov method***
  - A set of iterative techniques for performing various operations, such as approximately inverting a matrix or finding approximations to its eigenvectors or eigenvalues without using any operation other than matrix-vector products. Using this, we can compute Hessian in the form of:

$$H\boldsymbol{v} = \nabla_{\boldsymbol{x}} \left[\left(\nabla_{\boldsymbol{x}} f(\boldsymbol{x}) \right)^T \boldsymbol{v} \right]$$

Conclusion with Historical Remarks

- The core ideas behind feedforward networks (BackProp, gradient descent) have not changed since the 1980s.
- Most of the improvement in neural network performance from 1986 to Now can be attributed two factors: larger datasets and larger networks
- One of the algorithmic changes was replacement of means squared error with the cross-entropy family of loss functions and the idea of maximum likelihood, which less suffers from saturation and slow learning than using the mean squared error loss
- Another algorithmic change was the replacement of hidden sigmoid unit with ReLu and its variants. Why ReLu is better than non-linear ones is still of interest